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The prolongation structure of the inhomogeneous equation of the reaction–diffusion type

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Abstract
The inhomogeneous extension of equations of the reaction–diffusion type is investigated by means of the covariant prolongation structures theory. We construct the $sl(2, R) \times R(\rho(t))$ prolongation structure for an inhomogeneous equation of the reaction–diffusion type and give the corresponding AKNS-type equations and the Bäcklund transformation.

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1. Introduction

Equations of the reaction–diffusion type play a crucial role in many research areas. Based upon applications in biological systems and chemical autocatalysis, one of the reaction–diffusion types has received considerable attention [1],

$$\begin{aligned} &u_t - u_{xx} + 2u^2 v - 2ku = 0, \\ &v_t + v_{xx} - 2uv^2 + 2kv = 0. \end{aligned} \tag{1}$$

This equation also emerges in the gauge formulation of the (1+1)-dimensional gravity [2]. From the geometric equivalent point of view, the geometrical equivalent counterpart of (1) is the following modified Heisenberg ferromagnet (HF) equation [3, 4],

$$\mathbf{S}_t = \mathbf{S} \bar{\times} \mathbf{S}_{xx}, \tag{2}$$

where $\mathbf{S} = (s_1, s_2, s_3)$ with $\mathbf{S}^2 = s_1^2 + s_2^2 - s_3^2 = 1$ and $s_3 > 0$.

The prolongation structure theory proposed by Wahlquist and Estabrook (W–E) [5, 6] is a very useful tool in the analysis of (1+1)-dimensional integrable equations, such as the KDV equation, the higher order nonlinear Schrödinger equation and the sine-Gordon equation [7–12]. This theory can also be interpreted in terms of Cartan–Ehresmann connection from the viewpoint of differential geometry [13, 14]. By using W–E's prolongation structure theory, Alfinito *et al* [15] carried out the detailed integrable analysis of (1). It indicates that this equation allows an incomplete prolongation algebra admitting an infinite-dimensional realization of the Kac-Moody type. Note that the prolongation structure equations given by W–E are not manifestly covariant under the transformation of generators of the prolongation algebra. By means of the theory of nonlinear realization of connection proposed by Lu *et al* [16], Guo *et al* [17] proposed a covariant geometry theory for the prolongation structures of the nonlinear evolution equation. In this covariant prolongation structures theory, a set of fundamental equations for the prolongation structure was presented. Based upon this theory, the $SL(2,R)\times R(l)$ principal prolongation structure for AKNS systems has been constructed and a new set of infinite number of polynomial conservation currents corresponding to the nonlinearity of $SL(2,R)$ group manifold has also been presented [18]. The merit of this prolongation structures theory is that it is not only the covariant geometry theory, but also very helpful in obtaining the Bäcklund transformation and other properties of the nonlinear integrable system. In this paper, we shall investigate the inhomogeneous extension of equation (1) by using this covariant prolongation structures theory.

2. Covariant theory for prolongation structure of nonlinear evolution equation

Let us start with a short summary of the covariant theory for prolongation structure of NEE that will be useful in what follows. For a more detailed description we refer the reader to [16, 17, 19].

For a given (1+1)-dimensional NEE, we can give a set of equivalent first-order partial differential equations with the independent variables (x,t,u_l), $l=1,2,\ldots,n$. Then we can define a set of two forms α_i such that it constructs a closed ideal. When these two forms restricted on the solution manifold $U=\{x,t,u_l(x,t)\}$ become zero, we recover the original (1+1)-dimensional NEE. Now let us consider a principal bundle $P(N,G)$ and a bundle $E(N,Y,G,P)$ associated with P, where N is base space, G is structure group whose algebra is the prolongation algebra and Y is standard fibre. Define a local cross-section on E, $\sigma : N\to E$ and its covariant derivatives

$$\omega^i = \mathrm{d}y^i + \Gamma^i_\mu(X,y)\,\mathrm{d}x^\mu, \tag{3}$$

where $X=\{x^\mu\}=\{x,t,u_l\}$ and

$$\Gamma^i_\mu = \Gamma^a_\mu(X)\lambda^i_a(y), \tag{4}$$

in which $\Gamma^a_\mu(X)$ are the coefficients of the connection on P and $\lambda^i_a(y)$ is the coefficients of the generators of the prolongation algebra. We can introduce a connection induced from the nonlinear connection one form ω^i,

$$L^i_k = L^i_{k\mu}\,\mathrm{d}x^\mu = \left[\lambda^i_a(y)\frac{\partial\lambda^a_k}{\partial x^\mu} + C^a_{bc}\Gamma^b_\mu(x)\lambda^c_k(y)\lambda^i_a(y)\right]\mathrm{d}x^\mu, \tag{5}$$

which is linear under the coordinate transformations on Y and the transformations induced by the action of G on Y. By using the induced connection $L^i_{k\mu}$, we can define the following covariant exterior derivative,

$$D^*\omega^i = d\omega^i + L^i_j\wedge\omega^j = -\tfrac{1}{2}F^a_{\mu\nu}\lambda^i_a\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu + \tfrac{1}{2}M^i_{jk}\omega^j\wedge\omega^k, \tag{6}$$

$$D^*L^i_j = dL^i_j + L^i_k\wedge L^k_j = \tfrac{1}{2}K^i_{j\mu\nu}\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu, \tag{7}$$

where $F^a_{\mu\nu}$ and M^i_{jk} are the curvature coefficients on P and the torsion coefficients in the fibre space, respectively. They can be expressed as

$$F^a_{\mu\nu} = \left(\frac{\partial\Gamma^a_\mu}{\partial x^\nu} - \frac{\partial\Gamma^a_\nu}{\partial x^\mu} + C^a_{bc}\Gamma^b_\mu\Gamma^c_\nu\right), \tag{8}$$

$$M^i_{jk} = \lambda^a_j\frac{\partial\lambda^i_a}{\partial y^k} - \lambda^a_k\frac{\partial\lambda^i_a}{\partial y^j}, \tag{9}$$

$$K^i_{j\mu\nu} = \frac{\partial L^i_{j\nu}}{\partial x^\mu} - \frac{\partial L^i_{j\mu}}{\partial x^\nu} + L^i_{l\mu}L^l_{j\nu} - L^i_{l\nu}L^l_{j\mu}. \tag{10}$$

If we extend the closed ideal I on N to the closed ideal $I' = \{\alpha^i, \omega^j\}$ on E, we have

$$D^*\omega^i \subset I'. \tag{11}$$

Using (6) and the closed ideal condition, we obtain the following covariant fundamental equations determining the prolongation structure,

$$\tfrac{1}{2}F^a_{\mu\nu}\lambda^i_a\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu = f^i_\beta\alpha^\beta, \tag{12}$$

$$\tfrac{1}{2}M^i_{kl}\omega^k\wedge\omega^l = \eta^i_l\wedge\omega^l, \tag{13}$$

where f^i_β and η^i_l are the zero and one forms on N, respectively. It should be pointed out that we may completely determine the prolongation structure of a given nonlinear system when the solutions of one fundamental equation can be found.

3. The prolongation structure of the equation of the reaction–diffusion type

We have introduced the covariant theory for prolongation structure of nonlinear evolution equation in the previous section. Based upon this theory, we will discuss the corresponding prolongation structure for the inhomogeneous extension of equation (1) in this section.

To begin with, let us introduce the new items $(h(x,t)u)_x$ and $(g(x,t)v)_x$ into (1),

$$\begin{aligned} &u_t - u_{xx} + 2u^2v - 2ku - (hu)_x = 0,\\ &v_t + v_{xx} - 2uv^2 + 2kv - (gv)_x = 0, \end{aligned} \tag{14}$$

where the functions $h(x,t)$ and $g(x,t)$ need to be determined. Taking $u_x = p$ and $v_x = q$ as the new independent variables, we can define the following set of two forms in the six-dimensional space $N = \{x, t, u, v, p, q\}$,

$$\begin{aligned} \alpha_1 &= \mathrm{d}u\wedge\mathrm{d}t - p\,\mathrm{d}x\wedge\mathrm{d}t,\\ \alpha_2 &= \mathrm{d}v\wedge\mathrm{d}t - q\,\mathrm{d}x\wedge\mathrm{d}t,\\ \alpha_3 &= -\mathrm{d}p\wedge\mathrm{d}t - \mathrm{d}u\wedge\mathrm{d}x + (2u^2v - 2uk - h_xu - hp)\,\mathrm{d}x\wedge\mathrm{d}t,\\ \alpha_4 &= -\mathrm{d}q\wedge\mathrm{d}t - \mathrm{d}v\wedge\mathrm{d}x + (-2v^2u + 2vk - g_xv - gq)\,\mathrm{d}x\wedge\mathrm{d}t, \end{aligned} \tag{15}$$

such that they constitute a closed ideal I. When the above two forms restricted on the solution manifold $U = \{x, t, u(x,t), v(x,t), p(x,t), q(x,t)\}$ become zero, we recover equation (14).

According to the covariant prolongation structure theory, we extend the ideal I by adding to it a set of one forms,

$$\omega^i = \mathrm{d}y^i + \Gamma^a_\mu\lambda^i_a\,\mathrm{d}x^\mu, \tag{16}$$

where $\{x^\mu, \mu = 1, 2, \ldots, 6\} = \{x, t, u, v, p, q\}$. The closed condition of the extended ideal $I' = \{\alpha^i, \omega^j\}$ leads to the covariant fundamental equations. Substituting the above two forms α^i into fundamental equation (12) , we have

$$\begin{aligned} &F_{12}^i - pF_{23}^i - qF_{24}^i - 2u(uv-k)F_{25}^i - 2v(uv-k)F_{26}^i = 0, \\ &F_{15}^i = F_{16}^i = F_{34}^i = F_{35}^i = F_{36}^i = F_{45}^i = F_{46}^i = F_{56}^i = 0, \\ &F_{13}^i - F_{25}^i = 0, \qquad F_{14}^i + F_{26}^i = 0. \end{aligned} \tag{17}$$

Solving equation (17), we obtain the following connection coefficients on the $sl(2R) \times R(\rho)$ principal bundle,

$$\begin{aligned} &\Gamma_1^1 = -2\rho, \qquad \Gamma_1^2 = -u, \qquad \Gamma_1^3 = -v, \\ &\Gamma_2^1 = -2(uv - k - 2\rho^2 + h\rho), \qquad \Gamma_2^2 = -(p - 2\rho u + hu), \qquad \Gamma_2^3 = q + 2\rho v - gv, \end{aligned} \tag{18}$$

the other components are zero, and the parameter satisfies

$$\rho_t = \rho h_x, \qquad \rho_x = 0, \qquad h = g. \tag{19}$$

λ_a^i in (16) are the coefficients of the generators of the prolongation algebra $sl(2R)$. The commutation relation of $sl(2R)$ is given by

$$[X_1, X_2] = -X_2, \qquad [X_1, X_3] = X_3, \qquad [X_2, X_3] = -2X_1, \tag{20}$$

where

$$X_a = \lambda_a^i(y)\frac{\partial}{\partial y_i}, \qquad a = 1, 2, 3. \tag{21}$$

From (19), it is easy to see that the function h takes the following expression,

$$h = \phi(t)x + \psi(t). \tag{22}$$

It is noted that the parameter ρ in the $sl(2, R) \times R(\rho)$ prolongation structure of MKDV equation [17] is a constant. Due to the introduction of the functions $h(x, t)$ and $g(x, t)$, the parameter ρ depends on the time t in this paper. In fact, ρ can be regarded as the spectral parameter. We will see it in the following discussion.

Let us take a two-dimensional linear space as the prolongation space. A linear realization of the prolongation algebra can be written as

$$X_1 = \frac{1}{2}\left(y_2\frac{\partial}{\partial y_2} - y_1\frac{\partial}{\partial y_1}\right), \qquad X_2 = -y_2\frac{\partial}{\partial y_1}, \qquad X_3 = -y_1\frac{\partial}{\partial y_2}. \tag{23}$$

By requiring $\omega^i|_U = 0$ for (16), we obtain the AKNS-type equations

$$Y_x = -FY, \qquad Y_t = -GY, \tag{24}$$

where $Y = (y_1, y_2)^\top$,

$$F = \begin{bmatrix} \rho & u \\ v & -\rho \end{bmatrix}, \tag{25}$$

$$G = \begin{bmatrix} uv - k - 2\rho^2 + h\rho & u_x - 2\rho u + hu \\ -(v_x + 2\rho v) + hv & -uv + k + 2\rho^2 - h\rho \end{bmatrix}, \tag{26}$$

and the spectral parameter satisfies (19). When $h = 0$, the matrices F and G reduce to the results of (1) derived in [15].

Taking a one-dimensional space as the prolongation space, we have a nonlinear realization of the prolongation algebra

$$X_1 = y\frac{\partial}{\partial y}, \qquad X_2 = y^2\frac{\partial}{\partial y}, \qquad X_3 = -\frac{\partial}{\partial y}. \tag{27}$$

On the requirement $\omega^i|_U = 0$ for (16), the following Riccati equation can be obtained,

$$\begin{aligned} y_x &= uy^2 + 2\rho y - v, \\ y_t &= [u_x + (h - 2\rho)u]y^2 + 2[(uv - k) - 2\rho^2 + \rho h]y + v_x + (2\rho - h)v. \end{aligned} \tag{28}$$

Based upon the Riccati equation (28), we may get the Bäcklund transformation of (14),

$$\begin{aligned} (u + u')_x &= \mp(u - u')[4\rho^2 + (u + u')(v + v')]^{\frac{1}{2}}, \\ (v + v')_x &= \mp(v - v')[4\rho^2 + (u + u')(v + v')]^{\frac{1}{2}}, \end{aligned} \tag{29}$$

and

$$\begin{aligned} \mp(u + u')_t &= (u - u')_x[4\rho^2 + (u + u')(v + v')]^{\frac{1}{2}} \pm (u + u')(uv + u'v') \\ &\quad + h(u - u')[4\rho^2 + (u + u')(v + v')]^{\frac{1}{2}}, \\ \pm(v + v')_t &= (v - v')_x[4\rho^2 + (u + u')(v + v')]^{\frac{1}{2}} \pm (v + v')(uv + u'v') \\ &\quad - h(v - v')[4\rho^2 + (u + u')(v + v')]^{\frac{1}{2}}. \end{aligned} \tag{30}$$

It is known that the geometrical equivalent counterpart of (1) is the modified HF equation (2). As done in [4], one can easily prove that the geometrical equivalent counterpart of (14) is the inhomogeneous modified HF equation

$$\mathbf{S}_t = \mathbf{S} \bar{\times} \mathbf{S}_{xx} + h\mathbf{S}_x, \tag{31}$$

where the function h is given by (22). Its Lax representation is

$$\frac{\partial\varphi}{\partial x} = U\varphi, \qquad \frac{\partial\varphi}{\partial t} = V\varphi, \tag{32}$$

in which

$$\begin{aligned} U &= \sum_{i=1}^{3} \rho s_i \tau_i, \\ V &= \sum_{i=1}^{3} \rho(\mathbf{s} \bar{\times} \mathbf{s}_x)_i \tau_i + \sum_{i=1}^{3} (\rho h + \rho^2) s_i \tau_i, \end{aligned} \tag{33}$$

where the spectral parameter satisfies (19) and τ_i are the generator of $su(1, 1)$, i.e.,

$$\tau_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tau_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

4. Summary

We have investigated the inhomogeneous equation of the reaction–diffusion type by using the covariant prolongation structure theory. The $sl(2, R) \times R(\rho(t))$ prolongation structure for the equation of the reaction–diffusion type was constructed in this paper. For this covariant prolongation structure theory, the 2×2 AKNS inverse scattering equations and the corresponding Riccati equations can be easily obtained by taking the different prolongation space. It should be mentioned that a lot of questions remain to be understood for the time-dependent spectrum parameter. Some of them will be in the forthcoming publication.

The (2+1)-dimensional integrable equations are of general interest [20–23]. Recently the integrable (2+1)-dimensional modified HF models [22] have been investigated by using Morris's prolongation structure theory [24]. Through the motion of Minkowski space curves endowed with an additional spatial variable, the corresponding geometric equivalent (2+1)-dimensional integrable extensions of the reaction–diffusion equation have also been presented

$$\begin{aligned} &\hat{\psi}_t + \hat{\psi}_{xy} - \hat{\gamma}\hat{\psi} = 0, \\ &\hat{\phi}_t - \hat{\phi}_{xy} + \hat{\gamma}\hat{\phi} = 0, \\ &\hat{\gamma}_x = -\partial_y(\hat{\phi}\hat{\psi}). \end{aligned} \tag{34}$$

It should be mentioned that the more general (2+1)-dimensional reaction–diffusion equations can be written as [25],

$$\begin{aligned} u_t &= D_1 \triangle u + b_1 u^2 v + b_2 u v^2 + b_3 u + b_4 v + b_5, \\ v_t &= D_2 \triangle v + c_1 u^2 v + c_2 u v^2 + c_3 u + c_4 v + c_5, \end{aligned} \tag{35}$$

where $\triangle$ is the Laplace operator in two-dimensional orthonormal coordinates, D_1 and D_2 are the diffusion constants, b_i and $c_i, i = 1, 2, \ldots, 5$, are the coefficients. The covariant prolongation structure theory has been generalized to the case of higher dimensions [26], where self-dual Yang–Mills equations have been well discussed. Whether the covariant prolongation structure theory can be used to get the more integrable higher dimensional reaction–diffusion equations is under investigation.

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